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# Schwinger action principle via linear quantum canonical transformations

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**Abstract.** We have applied the Schwinger action principle to general one-dimensional (1D), time-dependent quadratic systems via linear quantum canonical transformations, which allowed us to simplify the problems to be solved by this method. We show that while using a suitable linear canonical transformation, we can considerably simplify the evaluation of the propagator of the studied system to that for a free particle. The efficiency and exactness of this method is verified in the case of the simple harmonic oscillator. This technique enables us to evaluate easily and immediately the propagator in some particular cases such as the damped harmonic oscillator, the harmonic oscillator with a time-dependent frequency, and the harmonic oscillator with time-dependent mass and frequency, and in this way the propagator of the forced damped harmonic oscillator is easily calculated without any approach.

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## 1 Introduction

Feynman and Schwinger, independently of each other, formulated an action principle in accordance with quantum mechanics; the former is functional whereas the latter is operatorial [1]. The path-integral method was extended to various physical fields and turned out to be an efficient calculation tool. Recent years have seen renewed interest in evaluating propagators by means of the Schwinger action principle [2, 3]. We mention the calculations by Urrutia and Hernendez to evaluate the propagators of the time-dependent forced harmonic oscillator and that of a charged harmonic oscillator in a constant magnetic field [4, 5] and using space-time transformation techniques by Nassar and Machado [6, 7]. The major difficulty in this technique is the solution of Heisenberg equations of motion, which are operatorial equations. Consequently, each case and each system deals with the application of a different method. This diversity of methods makes the Schwinger action principle as a calculation tool more complicated. To overcome this difficulty, we found that canonical transformations [8, 9] are, in general, adequate to simplify complicated problems and transform them into simple ones. We took these time-dependent quadratic systems as examples for their simplicity and their physical interest. In general, time-dependent oscillators have enormous applications in quantum physics such as in molecular physics, quantum chemistry, and quantum optics. For example, small vibrations of systems can be described using harmonic oscillators, and surrounding influences on these vibrations are included by considering time-dependent parameters in the Hamiltonian of a harmonic oscillator, such as the mass and the frequency. Although this oscillator model is taken as an approximate description of the vibration, it is a good approach for studying more complicated systems. Furthermore, in some cases, this becomes an exact model of system dynamics. For example, the ion in a Paul trap is described accurately by a harmonic oscillator with periodically time-dependent frequency [10]. For these reasons, for a long time, several attempts were made to solve exactly the Schrödinger equation using different methods. The invariant method proposed by Lewis–Riesenfeld [11] was intensely studied for these systems [12]. An appproach that uses the Noether theorem has been analyzed [13] and some time-dependent integrals of motion deduced [14].

Quantum canonical transformations that include both point transformations and time evolution are, in general, adequate applications for transforming the wave function for a system of interest into a simpler equation whose solution is known. The founding fathers of quantum mechanics defined quantum canonical transformations as transformations of the phase space variables that preserve the canonical commutation relations. Several of the familiar tools used to solve problems in quantum mechanics and

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quantum field theory are quantum canonical transformations, and only a few of them are unitary [15]. They play three essential roles: in evolution, for solving a theory, and in physical equivalence. They are used as practical tools for solving differential equations including Lie algebra methods, intertwining operators, and raising and lowering operators [16]. They have been used to solve the Schrödinger equation for different time-independent systems. An exact solution of the Schrödinger equation is possible for a limited number of potentials. If the system is a time-dependent one, an exact solution is generally not obvious. However, in time-dependent problems, quantum canonical transformations have not been used often without a harmonic oscillator with time-dependent frequency [17].

The present work is a successful attempt to investigate the Schwinger action principle by means of quantum canonical transformations to calculate exactly the propagator of a general 1D time-dependent quadratic system. Following a linear quantum canonical transformation, the system turns out to be equivalent to a free particle moving in one dimension. In the case of Hamiltonians with solely quadratic terms, we must solve a second-order homogeneous differential equation. For quadratic systems with additive $\hat{x}$ and $\hat{p}$ terms, we must solve a supplementary first-order differential system of two equations. The exactness of this result is checked by taking the particular case of a simple harmonic oscillator. The general result is then used to evaluate the propagator in some particular cases such as the damped harmonic oscillator, the harmonic oscillator with time-dependent frequency, the harmonic oscillator with time-dependent mass and frequency, and the forced damped harmonic oscillator.

Our method is compared with other studies in which the propagator was calculated using a path-integral method for the well-known forced harmonic oscillator [18] and in the case of the Calidora–Kanai oscillator [19].

The remainder of this article is organized as follows. In Sect. 2, we present a brief review of the Schwinger action principle. In Sect. 3, some general definitions of quantum canonical transformations are presented. Section 4 is devoted to the general method of calculations, and the general 1D time-dependent system is treated. In Sect. 5, the time-dependent quadratic harmonic oscillator is studied, and some particular cases are explicitly evaluated. In Sect. 6, we are concerned with the case of the forced damped harmonic oscillator with some explicitly particular cases. Some concluding remarks are given in the last section.

## 2 Formalism of Schwinger action principle

The Schwinger action principle expresses the variation of the propagator by means of the stationary variation of a Hermitian operator called the action of the system as follows:

$$\delta \langle x'', t'' \mid x', t' \rangle = \frac{i}{\hbar} \left\langle x'', t'' \middle| \delta\hat{W} \mid x', t' \right\rangle \tag{1}$$

where $x'$ and $x''$ are, respectively, the initial and final coordinates taken at the initial and final instants $t'$ and $t''$ and $\delta\hat{W}$is the variation of the action operator defined by

$$\hat{W} = \int_{t_1}^{t_2} \hat{L}\, \mathrm{d}t \tag{2}$$

where $\hat{L}$ is a Lagrangian operator of the physical system. In quantum mechanics, the phase space variables $(x, p)$ are considered to be operators $(\hat{x}, \hat{p})$ that act on functions in configuration space. However, they cannot be viewed as coordinates on the phase space in the usual sense. Consequently, the infinitesimal Hermitian operator $\delta\hat{W}$ is a function of these operators, and it takes the form

$$\delta\hat{W} = \hat{p}''\delta\hat{x}'' - \hat{p}'\delta\hat{x}' - \hat{H}''\delta t'' + \hat{H}'\delta t' \tag{3}$$

where $\hat{H}(\hat{p}, \hat{x})$ is the Hamiltonian of the system. The evaluation of $\delta\hat{W}$ requires solving operatorial Heisenberg equations of motion. We make use of the commutation properties of $\hat{x}''$ and $\hat{x}'$ to rearrange the operator $\delta\hat{W}$ so that the coordinate $\hat{x}''$ stands to the left of $\hat{x}'$. Then (1) can be written as

$$\delta \langle x'', t'' \mid x', t' \rangle = \frac{i}{\hbar} \delta w\left(x'', x', t'', t'\right) \langle \hat{x}'', t'' \mid \hat{x}', t' \rangle \tag{4}$$

where $\delta w\left(x'', x', t'', t'\right)$ is the variation of the well-ordered action. The integration of (4) gives the propagator

$$\langle x'', t'' \mid x', t' \rangle = C \exp\left(\frac{i}{\hbar} w\left(x'', x', t'', t'\right)\right) \tag{5}$$

which describes the probability amplitude for the transition of the physical system from position $x'$ at instant $t'$ to position $x''$ at instant $t''$. The coefficient $C$ here emerges from the requirement that

$$\lim_{t'' \longrightarrow t'} \langle x'', t'' \mid x', t' \rangle = \delta\left(x'' - x'\right) \tag{6}$$

## 3 Quantum canonical transformations

Due to noncommutativity, the strength of canonical transformations, which are a powerful tool of classical mechanics, has not been fully realized in quantum mechanics. One of the aims of developing classical canonical transformations is to transform the Hamiltonian for particular systems into a simpler one whose equations of motion can be solved. We will take advantage of this tool to evaluate propagators of quadratic systems by applying the Schwinger action principle.

Similar to the canonical transformations of classical Hamiltonian dynamics that preserve the Poisson brackets, quantum canonical transformations are defined as those transformations of the noncommuting phase space variables that preserve the Dirac brackets:

$$[\hat{x}, \hat{p}] = [\hat{Q}, \hat{P}] = i\hbar \tag{7}$$

Consequently, the new canonical variables $\left(\hat{Q}, \hat{P}\right)$ are Hermitian operators, and the eigenvectors of $\hat{Q}$ and $\hat{P}$ form a complete basis.

To make the quantum analogous to classical canonical transformations, let us consider the unitary operator $\hat{U}(t)$, which transforms the eigenvector of $\hat{x}$ into an eigenvector of $\hat{Q}$ as follows:

$$|\hat{x}\rangle = \hat{U}\left(\hat{Q}, \hat{P}, t\right)\left|\hat{Q}\right\rangle \tag{8}$$

The quantum generating function of the transformation is then defined in terms of the matrix elements of this unitary operator as

$$\exp\left[iF_1\left(x', Q'', t\right)/\hbar\right] \equiv \langle x' | Q'' \rangle_t = \langle x' | \hat{U}\left(\hat{Q}, t\right) | x'' \rangle \tag{9}$$

As in classical mechanics, other types of quantum generating functions are defined similarly as

$$\exp\left[iF_2\left(x', P'', t\right)/\hbar\right] \equiv \langle x' | P'' \rangle_t = \langle x' | \hat{U} | p'' \rangle \tag{10}$$

$$\exp\left[iF_3\left(p', Q'', t\right)/\hbar\right] \equiv \langle p' | Q'' \rangle_t = \langle p' | \hat{U} | x'' \rangle \tag{11}$$

$$\exp\left[iF_4\left(p', P'', t\right)/\hbar\right] \equiv \langle p' | P'' \rangle_t = \langle p' | \hat{U} | p'' \rangle \tag{12}$$

Finally, the transformation relations between the old and new variable representations are written, similarly to the classical case, in terms of the well-ordered generating function $\hat{F}_1$ as follows:

$$\hat{p} = \frac{\partial \hat{F}_1\left(\hat{x}, \hat{Q}, t\right)}{\partial \hat{x}}, \qquad \hat{P} = -\frac{\partial \hat{F}_1\left(\hat{x}, \hat{Q}, t\right)}{\partial \hat{Q}} \tag{13}$$

We obtain similar relations for the generating function $\hat{F}_2$ that will be used in the next section:

$$\hat{p} = \frac{\partial \hat{F}_2\left(\hat{x}, \hat{P}, t\right)}{\partial \hat{x}}, \qquad \hat{Q} = \frac{\partial \hat{F}_2\left(\hat{x}, \hat{P}, t\right)}{\partial \hat{P}} \tag{14}$$

Taking into account the symmetrization rule, the relation between the two quantum generating functions is given as

$$\hat{F}_1\left(\hat{x}, \hat{Q}, t\right) = -\frac{1}{2}\left(\hat{Q}\hat{P} + \hat{P}\hat{Q}\right) + \hat{F}_2\left(\hat{x}, \hat{P}, t\right) \tag{15}$$

The new transformed Hamiltonian that represents the time evolution of the new coordinate representation is expressed in terms of the generating function as

$$\hat{\tilde{H}} = \hat{H} + \frac{\partial \hat{F}_2}{\partial t} \tag{16}$$

In this case, the variation of the action operator can be expressed in the new representation as

$$\begin{aligned}\delta \hat{W}\left(\hat{P}, \hat{Q}, t\right) = {} & \hat{P}''\delta\hat{Q}'' - \hat{\tilde{H}}''\delta t'' - \hat{P}'\delta\hat{Q}' \\ & + \hat{\tilde{H}}'\delta t' + \delta\left(\hat{F}_1'' - \hat{F}_1'\right)\end{aligned} \tag{17}$$

Now we will express the propagator in terms of the new coordinates. Equation (8) enables us to write

$$\begin{aligned}\langle x'', t'' | x', t' \rangle = {} & \langle Q'', \tau'' | \\ & U\left(\hat{Q}'', t''\right) U\left(\hat{Q}', t'\right) | Q', \tau' \rangle\end{aligned}$$

If we consider $K\left(\hat{Q}'', \hat{Q}', t'', t'\right) = U\left(\hat{Q}'', t''\right) U\left(\hat{Q}', t'\right)$, then

$$\begin{aligned}\langle x'', t'' | x', t' \rangle = {} & K\left(Q'', Q', \tau'', \tau'\right) \\ & \langle Q'', \tau'' | Q', \tau' \rangle\end{aligned} \tag{18}$$

Hence, (1) becomes in the new space-time representation

$$\begin{aligned}\delta\langle Q'', \tau'' | Q', \tau' \rangle = {} & \frac{i}{\hbar}\Big\langle Q'', \tau'' \Big| \\ & \delta\hat{W}\left(\hat{Q}'', \hat{Q}', \tau'', \tau'\right)\Big| \\ & Q', \tau' \Big\rangle - \Big\langle Q'', \tau'' \Big| \\ & Q', \tau' \Big\rangle \delta \ln K\end{aligned} \tag{19}$$

We rearrange the operator $\delta\hat{W}\left(\hat{Q}'', \hat{Q}', \tau'', \tau'\right)$ so that $q''$ is on the left and $q'$ on the right to obtain the well-ordered variation of the action operator $\delta w\left(Q', Q'', \tau', \tau''\right)$. Following integration, the propagator takes the following form in the new space-time system:

$$\langle Q'', \tau'' \mid Q', \tau' \rangle = AK^{-1}\exp\left(\frac{i}{\hbar}w\left(Q', Q'', \tau', \tau''\right)\right) \tag{20}$$

where $A$ is determined from the normalization condition (6).

## 4 Propagator of general 1D time-dependent quadratic system via quantum canonical transformations

### 4.1 The Hamiltonian and the action operator variation in the new space-time representation

In this section, we will evaluate the propagator of the general 1D quadratic system by means of quantum canonical transformations, without passing by the resolution of Heisenberg equations of motion and without solving complicated differential operatorial equations. The Hamiltonian operator of the system is given by the expression

$$\begin{aligned}\hat{H} = {} & g_1(t)\hat{p}^2 + g_2(t)\hat{x}^2 + g_3(t)\hat{x}\hat{p} \\ & + g_4(t)\hat{p}\hat{x} + g_5(t)\hat{p} + g_6(t)\hat{x} + g_7(t)\end{aligned} \tag{21}$$

where $g_i(t)$ are time-dependent functions. Now consider the following canonical transformation:

$$\begin{cases}\hat{x} = \left(2mg_1\right)^{1/2} f_1(t)\hat{Q} + f_2(t) \\ \hat{p} = \dfrac{\hat{P}}{\left(2mg_1\right)^{1/2} f_1(t)} + f_3(t)\end{cases} \tag{22}$$

with $f_i(t)$ an arbitrary time-dependent function and $m$ the mass of the particle. Equations (13) and (14) enable us to find the following generating functions:

$$\begin{cases} \hat{\not{F}}_2 = \dfrac{\left(\hat{x}\hat{P}+\hat{P}\hat{x}\right)}{2\left(2mg_1\right)^{1/2} f_1(t)} + f_3(t)\,\hat{x} - \dfrac{f_2(t)\,\hat{P}}{\left(2mg_1\right)^{1/2} f_1(t)} \\ \quad = \frac{1}{2}\left(\hat{Q}\hat{P}+\hat{P}\hat{Q}\right) + \left(2mg_1\right)^{1/2} f_3(t)\, f_1(t)\,\hat{Q} \\ \quad + f_3(t)\, f_2(t) \\ \hat{\not{F}}_1 = \left(2mg_1\right)^{1/2} f_3(t)\, f_1(t)\,\hat{Q} + f_3(t)\, f_2(t) \end{cases} \tag{23}$$

Hence the Hamiltonian becomes, after using (16),

$$\begin{aligned} \hat{\hat{H}}\left(\hat{P},\hat{Q},t\right) =& \frac{\hat{P}^2}{2mf_1^2} + 2mg_1g_2f_1^2\hat{Q}^2 \\ &+ \frac{1}{2}\left(g_3+g_4-\frac{\dot{f}_1}{f_1}-\frac{\dot{g}_1}{2g_1}\right)\left(\hat{P}\hat{Q}+\hat{Q}\hat{P}\right) \\ &+ \frac{\left(2g_1f_3+g_3f_2+g_4f_2+g_5-\dot{f}_2\right)}{\left(2mg_1f_1^2\right)^{1/2}}\hat{P} \\ &+ \left(2mg_1f_1^2\right)^{1/2} \\ &\quad \times\left(2g_2f_2+g_3f_3+g_4f_3+g_6+\dot{f}_3\right)\hat{Q} \\ &+ g_1f_3^2+g_2f_2^2+g_3f_2f_3+g_4f_2f_3+g_5f_3 \\ &+ g_6f_2+g_7+f_2\dot{f}_3+\frac{i\hbar}{2}\left(g_3-g_4\right) \end{aligned} \tag{24}$$

where $\dot{f}_i = \dfrac{\partial f_i}{\partial t}$. Now to drop the $\hat{Q}$ and $\hat{P}$ terms, we will nullify their coefficients, that is, we will solve the following differential equation system:

$$\begin{cases} \dot{f}_2 - f_2\left(g_3+g_4\right) - 2g_1f_3 = g_5 \\ \dot{f}_3 + 2g_2f_2 + f_3\left(g_3+g_4\right) = -g_6 \end{cases} \tag{25}$$

whose solution depends on the functions $g_i(t)$. Consequently, the Hamiltonian and the action operator variation are written in the new coordinate representation as

$$\begin{cases} \hat{\hat{H}}\left(\hat{P},\hat{Q},t\right) = \dfrac{\hat{P}^2}{2mf_1^2} + 2mg_1g_2f_1^2\hat{Q}^2 \\ \qquad + \frac{1}{2}\left(g_3+g_4-\dfrac{\dot{f}_1}{f_1}-\dfrac{\dot{g}_1}{2g_1}\right) \\ \qquad \times\left(\hat{P}\hat{Q}+\hat{Q}\hat{P}\right) + \Gamma(t) \\ \delta\hat{W} = \hat{P}''\delta\hat{Q}'' - \hat{\hat{H}}''\delta t'' - \hat{P}'\delta\hat{Q}' + \hat{\hat{H}}'\delta t' \\ \quad + \delta\Big[\left(2mg_1''\right)^{1/2} f_1'' f_3''\hat{Q}'' + f_2''f_3'' \\ \quad - \left(2mg_1'\right)^{1/2} f_1'f_3'\hat{Q}' - f_2'f_3'\Big] \end{cases} \tag{26}$$

and

$$\begin{aligned} \Gamma(t) =& g_1f_3^2 + g_2f_2^2 + \left(g_3+g_4\right)f_2f_3 + g_5f_3 \\ &+ g_6f_2 + g_7 + f_2\dot{f}_3 + \frac{i\hbar}{2}\left(g_3-g_4\right). \end{aligned} \tag{27}$$

We will now introduce the following canonical transformation:

$$\begin{cases} \hat{Q} = \hat{Q} \\ \hat{\hat{P}} = \hat{P} + mf_1^2\left(g_3+g_4-\dfrac{\dot{f}_1}{f_1}-\dfrac{\dot{g}_1}{2g_1}\right)\hat{Q} \end{cases} \tag{28}$$

With regard to (14) and (15), the generating functions for this transformation are thus

$$\begin{cases} \hat{\hat{\not{F}}}_2 = \frac{1}{2}\left(\hat{Q}\hat{\hat{P}}+\hat{\hat{P}}\hat{Q}\right) - \frac{1}{2}mf_1^2\left(g_3+g_4-\dfrac{\dot{f}_1}{f_1}-\dfrac{\dot{g}_1}{2g_1}\right)\hat{Q}^2 \\ \hat{\hat{F}}_1 = -\frac{1}{2}mf_1^2\left(g_3+g_4-\dfrac{\dot{f}_1}{f_1}-\dfrac{\dot{g}_1}{2g_1}\right)\hat{Q}^2 \end{cases} \tag{29}$$

Then we get from (16) the following Hamiltonian:

$$\hat{\hat{H}}\left(\hat{\hat{P}},\hat{Q},t\right) = \frac{\hat{\hat{P}}}{2mf_1^2} + \frac{m}{2f_1^2}\Omega^2(t)\,\hat{Q}^2 + \Gamma(t) \tag{30}$$

where

$$\begin{aligned} \Omega^2(t) =& f_1^3\Bigg(\ddot{f}_1 + f_1\bigg(\frac{\ddot{g}_1}{2g_1} - \frac{3\dot{g}_1^2}{4g_1^2} + 4g_2g_1 \\ &- \left(g_3+g_4\right)^2 - \left(\dot{g}_3+\dot{g}_4\right) + \frac{\dot{g}_1}{g_1}\left(g_3+g_4\right)\bigg)\Bigg) \end{aligned} \tag{31}$$

We will now make the following temporal transformation:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{f_1^2} \tag{32}$$

and we write the Hamiltonian as

$$\hat{\not{H}}\left(\hat{\hat{P}},\hat{Q},\tau\right) = f_1^2\,\hat{\hat{H}}\left(\hat{\hat{P}},\hat{Q},t\right) \tag{33}$$

Finally, the Hamiltonian is written as

$$\hat{\not{H}} = \frac{\hat{\hat{P}}^2}{2m} + \frac{1}{2}m\Omega^2\hat{Q}^2 + f_1^2\Gamma \tag{34}$$

and the variations of the action operator are expressed in the new representation as

$$\begin{aligned} \delta\hat{W} =& \hat{\hat{P}}''\delta\hat{Q}'' - \hat{\not{H}}''\delta\tau'' - \hat{\hat{P}}'\delta\hat{Q}' + \hat{\not{H}}'\delta\tau' \\ &+ \delta\Big[\left(2mg_1''\right)^{1/2} f_1''f_3''\hat{Q}'' + f_2''f_3'' - \left(2mg_1'\right)^{1/2} \\ &\quad \times f_1'f_3'\hat{Q}' - f_2'f_3' \\ &- \frac{mf_1''}{2}\left(f_1''\left(\left(g_3''+g_4''\right) - \frac{\dot{g}_1''}{2g_1''}\right) - \dot{f}_1''\right)\hat{Q}''^2 \\ &+ \frac{mf_1'}{2}\left(f_1'\left(\left(g_3'+g_4'\right) - \frac{\dot{g}_1'}{2g_1'}\right) - \dot{f}_1'\right)\hat{Q}'^2\Big]. \end{aligned} \tag{35}$$

Let us consider in (31) $\Omega(t) = 0$, that is, we wish to solve the following second-order homogeneous differential equation:

$$\ddot{f}_1 + \zeta(t)\, f_1 = 0 \tag{36}$$

where

$$\zeta(t) = \frac{\ddot{g}_1}{2g_1} - \frac{3\dot{g}_1^2}{4g_1^2} + 4g_2g_1 - (g_3+g_4)^2 - (\dot{g}_3+\dot{g}_4) + \frac{\dot{g}_1}{g_1}(g_3+g_4) \tag{37}$$

It is obvious that the solution of this equation depends on the expressions of the coefficients $g_i(t)$, i.e., on the nature of the studied system. We obtain the Hamiltonian of a free particle with an additional time-dependent term:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + f_1^2\Gamma \quad \text{and} \quad \hat{\mathcal{H}}_0 = \frac{\hat{\mathcal{P}}^2}{2m} \tag{38}$$

However, we can write the variation of the action operator as

$$\begin{aligned}\delta\hat{W} =& \hat{\mathcal{P}}''\delta\hat{Q}'' - \hat{\mathcal{H}}_0''\delta\tau'' - \hat{\mathcal{P}}''\delta\hat{Q}' + \hat{\mathcal{H}}_0'\delta\tau' \\ &+ \delta\Big[(2mg_1'')^{1/2} f_1''f_3''\hat{Q}'' + f_2''f_3'' \\ &- (2mg_1')^{1/2} f_1'f_3'\hat{Q}' - f_2'f_3' \\ &- \frac{mf_1''}{2}\left(f_1''\left((g_3''+g_4'') - \frac{\dot{g}_1''}{2g_1''}\right) - \dot{f}_1''\right)\hat{Q}''^2 \\ &+ \frac{mf_1'}{2}\left(f_1'\left((g_3'+g_4') - \frac{\dot{g}_1'}{2g_1'}\right) - \dot{f}_1'\right)\hat{Q}'^2 - \Lambda\Big]\end{aligned} \tag{39}$$

where $\Lambda = \int_{\tau'}^{\tau''} f_1^2\Gamma\,\mathrm{d}\tau = \int_{t'}^{t''}\Gamma\,\mathrm{d}t$ and $\Gamma$ verifies (27).

We recall now the Schwinger action principle, and (4), (20), and (39) enable us to write the propagator of the system immediately by means of the propagator of the free particle as

$$\begin{aligned}&\langle Q'',\tau''|Q',\tau'\rangle \\ &= AK^{-1}\langle Q'',\tau''|Q',\tau'\rangle_{FP} \\ &\times\exp\frac{i}{\hbar}\Big[(2mg_1'')^{1/2}f_1''f_3''\hat{Q}'' + f_2''f_3'' - (2mg_1')^{1/2}f_1'f_3'\hat{Q}' \\ &- f_2'f_3' - \frac{mf_1''}{2}\left(f_1''\left((g_3''+g_4'') - \frac{\dot{g}_1''}{2g_1''}\right) - \dot{f}_1''\right)\hat{Q}''^2 \\ &+ \frac{mf_1'}{2}\left(f_1'\left((g_3'+g_4') - \frac{\dot{g}_1'}{2g_1'}\right) - \dot{f}_1'\right)\hat{Q}'^2 - \Lambda\Big]\end{aligned} \tag{40}$$

Recognize that $\langle Q'',\tau''|Q',\tau'\rangle_{FP}$ is the well-known free-particle propagator

$$\langle Q'',\tau''|Q',\tau'\rangle_{FP} = \left(\frac{2\pi i\hbar(\tau''-\tau')}{m}\right)^{-1/2} \times\exp\left(\frac{im}{2\hbar}\frac{(Q''-Q')^2}{(\tau''-\tau')}\right) \tag{41}$$

To determine coefficient $K$ in propagator (40), the unitary operator in (8) should be calculated, but we are not going to take this step, as in our work there is no need to know the unitary operator of the canonical transformation. Expression (18) enables us to drop coefficient $K$ and to turn the propagator to the former representation by using the canonical transformation (22) and the temporal transformation (32), which leads to

$$\tau'' - \tau' = \int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2} \tag{42}$$

We find

$$\begin{aligned}\langle x'',t''|x',t'\rangle =& A\left(\frac{2\pi}{m}i\hbar\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2}\right)^{-1/2} \\ &\times\exp\left(\frac{i}{4\hbar}\left(\frac{(x''-f_2'')}{g_1''^{1/2}f_1''} - \frac{(x'-f_2')}{g_1'^{1/2}f_1'}\right)^2\Big/\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2}\right) \\ &\times\exp\Bigg(\frac{i}{\hbar}\Bigg(f_3''x'' - f_3'x' - \frac{(x''-f_2'')^2}{8g_1''} \\ &\times\left(2(g_3''+g_4'') - \frac{\dot{g}_1''}{g_1''} - \frac{2\dot{f}_1''}{f_1''}\right) \\ &+\frac{(x'-f_2')^2}{8g_1'}\left(2(g_3'+g_4') - \frac{\dot{g}_1'}{g_1'} - \frac{2\dot{f}_1'}{f_1'}\right) - \Lambda\Bigg)\Bigg)\end{aligned} \tag{43}$$

After using the expression

$$\lim_{\varepsilon\longrightarrow 0}(\varepsilon\pi)^{-1/2}\exp\left(-\frac{(x''-x')^2}{\varepsilon}\right) = \delta(x''-x') \tag{44}$$

and the properties of the function delta, the normalization condition (6) enables us to calculate coefficient $A$. We obtain, finally, the propagator of the system:

$$\begin{aligned}\langle x'',t''|x',t'\rangle =& \left(4\pi i\hbar g_1'^{1/2}g_1''^{1/2}f_1''f_1'\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2}\right)^{-1/2} \\ &\times\exp\left(\frac{i}{4\hbar}\left(\frac{(x''-f_2'')}{g_1''^{1/2}f_1''} - \frac{(x'-f_2')}{g_1'^{1/2}f_1'}\right)^2\Big/\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2}\right) \\ &\times\exp\Bigg(\frac{i}{\hbar}\Bigg(f_3''x'' - f_3'x' - \frac{(x''-f_2'')^2}{8g_1''} \\ &\times\left(2(g_3''+g_4'') - \frac{\dot{g}_1''}{g_1''} - \frac{2\dot{f}_1''}{f_1''}\right) + \frac{(x'-f_2')^2}{8g_1'} \\ &\times\left(2(g_3'+g_4') - \frac{\dot{g}_1'}{g_1'} - \frac{2\dot{f}_1'}{f_1'}\right) - \Lambda\Bigg)\Bigg)\end{aligned} \tag{45}$$

where $\Lambda = \int_{t'}^{t''}\Gamma\,\mathrm{d}t$, $\Gamma$ satisfies (27) and the functions $f_i(t)$ are the solutions of differential equations (36) and (25). Equation (45) represents the exact propagator of the general 1D time-dependent system determined by the generalized Hamiltonian written in (21). This result constitutes the main result of this paper, and it includes the following systems: free particle, quasifree particle, harmonic oscillator, damped harmonic oscillator, harmonic oscillator with a time-dependent frequency, harmonic oscillator with a time-dependent mass and frequency, forced harmonic oscillator, forced damped harmonic oscillator, etc..

## 5 Time-dependent quadratic harmonic oscillator

In the case of the general 1D time-dependent quadratic system interpreted above, the role of the functions $f_2(t)$ and $f_3(t)$ in the canonical transformation (22) was to drop the additional $\hat{x}$ and $\hat{p}$ terms. If we consider the case of the 1D time-dependent quadratic system with the following Hamiltonian:

$$\hat{H} = g_1(t)\hat{p}^2 + g_2(t)\hat{x}^2 + g_3(t)\hat{x}\hat{p} + g_4(t)\hat{p}\hat{x} \tag{46}$$

that is to say, in Hamiltonian (21) we take

$$g_5(t) = g_6(t) = g_7(t) = 0 \tag{47}$$

in this case, we will perform the following canonical transformation:

$$\begin{cases} \hat{x} = (2mg_1)^{1/2} f_1(t)\hat{Q} \\ \hat{p} = \dfrac{\hat{P}}{(2mg_1)^{1/2} f_1(t)} \end{cases} \tag{48}$$

and consider

$$f_2(t) = f_3(t) = 0 \tag{49}$$

Not to repeat the same steps as in the previous section, we are going to use immediately the result obtained in (45) after making the substitutions in (47) and (49). The propagator of the 1D quadratic system is then expressed as

$$\begin{aligned} \langle x'', t'' | x', t' \rangle &= \left( 4\pi i\hbar {g_1'}^{1/2} {g_1''}^{1/2} f_1'' f_1' \int_{t'}^{t''} \frac{dt}{f_1^2} \right)^{-1/2} \\ &\times \exp\left( \frac{i}{4\hbar} \left( \frac{x''}{{g_1''}^{1/2} f_1''} - \frac{x'}{{g_1'}^{1/2} f_1'} \right)^2 \Big/ \int_{t'}^{t''} \frac{dt}{f_1^2} \right) \\ &\times \exp\left( \frac{i}{\hbar} \left( -\frac{x''^2}{8g_1''} \left( 2(g_3'' + g_4'') - \frac{\dot{g}_1''}{g_1''} - \frac{2\dot{f}_1''}{f_1''} \right) \right.\right. \\ &\left.\left. + \frac{x'^2}{8g_1'} \left( 2(g_3' + g_4') - \frac{\dot{g}_1'}{g_1'} - \frac{2\dot{f}_1'}{f_1'} \right) - \int_{t'}^{t''} \frac{i\hbar(g_3 - g_4)}{2} dt \right) \right) \end{aligned} \tag{50}$$

where $f_1(t)$ is the solution of differential equation (36).

Before starting the applications of this formalism, it is significant to notice that the solution of the propagator (45) and (50) is dependent on the solvability of differential equations (25) and (36), which are linear. In consequence, their general solutions are a combination of independent solutions but depend on the arbitrary constants of integration. It is remarkable that the propagator, which represents a kernel of the physical system, will be independent of the arbitrary choice of these constants. In such a case, we have an invariance property relative to this choice of constants, and we call this property the "gauge invariance". In the following applications this invariance is demonstrated and easy to verify.

### 5.1 Applications

#### 5.1.1 The simple harmonic oscillator

We can verify the efficiency of the Schwinger principle via canonical transformations in a simple application: the case of a simple harmonic oscillator with constant frequency $\omega$ whose propagator is already known. The Hamiltonian operator of the system takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{51}$$

However, the $g_i(t)$ functions take the following values:

$$g_1(t) = \frac{1}{2m}, \quad g_2(t) = \frac{1}{2}m\omega^2, \quad g_3(t) = g_4(t) = 0 \tag{52}$$

Now we must pick a suitable time-dependent function to use it in a canonical transformation. Let us return to differential equation (36), which turns in our case into a second-order homogeneous differential equation with a constant coefficient:

$$\ddot{f}_1 + \omega^2 f_1 = 0 \tag{53}$$

The solution of this equation is $f_1 = C_1 \cos\omega t + C_2 \sin\omega t$. We substitute it into (50) and obtain the propagator of a simple harmonic oscillator taking its known form:

$$\begin{aligned} \langle x'', t'' | x', t' \rangle &= \left[ \frac{2\pi i\hbar \sin\omega(t'' - t')}{m\omega} \right]^{-1/2} \\ &\times \exp\left[ \frac{im\omega}{2\hbar} \cot\left[\omega(t'' - t')\right] \left( x''^2 + x'^2 - \frac{2x''x'}{\cos\omega(t'' - t')} \right) \right] \end{aligned} \tag{54}$$

As expected, this last expression is independent of the choice of the constants.

#### 5.1.2 Harmonic oscillator with a time-dependent frequency

In this section, we will evaluate the propagator of a harmonic oscillator with a time-dependent frequency by means of quantum canonical transformations. The Hamiltonian operator of the system is given by the expression

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{x}^2 \tag{55}$$

where $\omega(t)$ is a time-dependent frequency. In this case, the $g_i(t)$ functions are written as

$$g_1(t) = \frac{1}{2m}, \quad g_2(t) = \frac{1}{2}m\omega^2(t), \quad g_3(t) = g_4(t) = 0 \tag{56}$$

When we substitute these functions into differential equation (36), we obtain

$$\ddot{f}_1+\omega^2(t)f_1=0 \tag{57}$$

The solution of this equation depends on the expression of the frequency $\omega(t)$. In this case, the propagator of the harmonic oscillator with a time-dependent frequency is expressed as

$$\begin{aligned}\langle x'',t''|x',t'\rangle &= \left(\frac{2\pi i\hbar}{m}f_1''f_1'\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2}\right)^{-1/2}\\ &\times\exp\left(\frac{im}{2\hbar}\left(\frac{x''}{f_1''}-\frac{x'}{f_1'}\right)^2\Big/\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2}+\frac{\dot{f}_1''}{f_1''}x''^2-\frac{\dot{f}_1'}{f_1'}x'^2\right)\end{aligned} \tag{58}$$

When we consider in (31) $\Omega(t)=f_1^3\left(\ddot{f}_1+\omega^2(t)f_1\right)=\omega_0$, where $\omega_0$ is a constant, that is, when we transform the system not into a free-particle but a simple harmonic oscillator, we obtain

$$\begin{aligned}\langle x'',t''|x',t'\rangle &= \left[\frac{2\pi i\hbar f_1''f_1'\sin\left(\int_{t'}^{t''}\frac{\omega_0\,\mathrm{d}t}{f_1^2}\right)}{m\omega_0}\right]^{-1/2}\\ &\times\exp\left[\frac{im\omega_0}{2\hbar}\cot\left(\int_{t'}^{t''}\frac{\omega_0\,\mathrm{d}t}{f_1^2}\right)\right.\\ &\times\left.\left(\frac{x''^2}{f_1''^2}+\frac{x'^2}{f_1'^2}-\frac{2x''x'}{f_1''f_1'\cos\left(\int_{t'}^{t''}\frac{\omega_0\,\mathrm{d}t}{f_1^2}\right)}\right)\right]\\ &\times\exp\frac{im}{2\hbar}\left(\frac{\dot{f}_1''}{f_1''}x''^2-\frac{\dot{f}_1'}{f_1'}x'^2\right)\end{aligned} \tag{59}$$

which is exactly the same result obtained in [9] by means of a path-integral method in the case of a harmonic oscillator with a time-dependent frequency.

### 5.1.3 Harmonic oscillator with time-dependent mass and frequency

We will consider the following Hamiltonian of a harmonic oscillator with time-dependent mass and frequency:

$$\hat{H}=\frac{\hat{p}^2}{2mh_1(t)}+\frac{1}{2}m\omega^2h_2(t)\hat{x}^2 \tag{60}$$

where $h_i(t)$ are functions depending on time and $m$ and $\omega$ are constants. For this system, the $g_i(t)$ functions take the forms

$$g_1(t)=\frac{1}{2mh_1(t)},\quad g_2(t)=\frac{1}{2}m\omega^2h_2(t),\quad g_3(t)=g_4(t)=0 \tag{61}$$

However, differential equation (36) takes the form

$$\ddot{f}_1+\left(-\frac{\ddot{h}_1(t)}{2h_1(t)}+\frac{\dot{h}_1^2(t)}{4h_1^2(t)}+\omega^2\frac{h_2(t)}{h_1(t)}\right)f_1=0 \tag{62}$$

which is more complicated than the previous ones. We recall the general form (50) to write, finally, the propagator of the harmonic oscillator with time-dependent mass and frequency as

$$\begin{aligned}\langle x'',t''|x',t'\rangle &= \left(\frac{2\pi i\hbar}{m}f_1'f_1''h_1'^{-1/2}h_1''^{-1/2}\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2}\right)^{-1/2}\\ &\times\exp\left(\frac{i}{4\hbar}\left(\left(\frac{h_1''^{-1/2}x''}{f_1''g(t'')^{1/2}}-\frac{h_1'^{-1/2}x'}{f_1'g(t')^{1/2}}\right)^2\Big/\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2(t)}\right.\right.\\ &\left.\left.+\left(\frac{\dot{f}_1''}{f_1''}-\frac{\dot{h}_1''}{2h_1''}\right)h_1''x''^2-\left(\frac{\dot{f}_1'}{f_1'}-\frac{\dot{h}_1'}{2h_1'}\right)h_1'x'^2\right)\right)\end{aligned} \tag{63}$$

### 5.1.4 Damped harmonic oscillator

The Hamiltonian of a damped harmonic oscillator is given by the following expression:

$$\hat{H}=\frac{\hat{p}^2}{2m\exp 2\xi(t)}+\frac{1}{2}m\omega^2\exp 2\xi(t)\hat{x}^2 \tag{64}$$

We can use immediately the result (63) related to the harmonic oscillator with a time-dependent mass and frequency to obtain the propagator of a damped harmonic by making the following substitutions:

$$h_1(t)=h_2(t)=\exp 2\xi(t) \tag{65}$$

The propagator is then expressed as

$$\begin{aligned}\langle x'',t''|x',t'\rangle &= \left(\frac{2\pi i\hbar f'f''}{m\exp\left(\xi(t'')+\xi(t')\right)}\int_{t'}^{t''}\frac{\mathrm{d}t}{f^2(t)}\right)^{-1/2}\\ &\times\exp\left(\frac{im}{2\hbar}\left(\left(\frac{x''\exp\left(\xi(t'')\right)}{f''}-\frac{x'\exp\left(\xi(t')\right)}{f'}\right)^2\Big/\int_{t'}^{t''}\frac{\mathrm{d}t}{f^2(t)}\right.\right.\\ &+\left(\frac{\dot{f}''}{f''}-\dot{\xi}(t'')\right)x''^2\exp\left(2\xi(t'')\right)\\ &\left.\left.-\left(\frac{\dot{f}'}{f'}-\dot{\xi}(t')\right)x'^2\exp\left(2\xi(t')\right)\right)\right)\end{aligned} \tag{66}$$

so that the function $f$ is the solution of the following differential equation:

$$\ddot{f}+\left(-\ddot{\xi}(t)-\dot{\xi}^2(t)+\omega^2\right)f=0 \tag{67}$$

It is clear here also that the results (58), (59), (63), and (66) are "gauge invariant", i.e., independent of the choice of the constants.

### 5.1.5 Calidora–Kanai oscillator

The Calidora–Kanai [15] is a particular case of the damped harmonic oscillator. Its Hamiltonian is expressed as

$$\hat{H} = \frac{\hat{p}^2}{2m\exp(2\alpha t)} + \frac{1}{2}m\exp(2\alpha t)\omega^2\hat{x}^2 \tag{68}$$

However, we will consider the function $\xi(t)$ in (66) to be a linear one, such as $\xi(t) = \alpha t$. Thus, differential equation (67) turns into

$$\ddot{f} + \left(\omega^2 - \alpha^2\right) f = 0 \tag{69}$$

The solution is then

$$f = C_1 \cos\left(\sqrt{\omega^2-\alpha^2}t\right) + C_2 \sin\left(\sqrt{\omega^2-\alpha^2}t\right) \tag{70}$$

We substitute it into (66) and obtain the propagator of a Calidora–Kanai oscillator:

$$\begin{aligned}\langle x'', t'' | x', t' \rangle &= \left(\frac{2\pi i\hbar \sin\left(\sqrt{\omega^2-\alpha^2}(t''-t')\right)}{m\exp(\alpha(t''+t'))\sqrt{\omega^2-\alpha^2}}\right)^{-1/2} \\ &\times \exp\left(\frac{im}{2\hbar}\left(\sqrt{\omega^2-\alpha^2}\cot\left[\sqrt{\omega^2-\alpha^2}(t''-t')\right]\right.\right. \\ &\times \left(x''^2\exp(2\alpha t'') + x'^2\exp(2\alpha t')\right. \\ &\left. - \frac{2x''x'\exp\left(\alpha(t''+t')\right)}{\cos\left[\sqrt{\omega^2-\alpha^2}(t''-t')\right]}\right) \\ &\left.\left. - \alpha\left(\exp(2\alpha t'')x''^2 - \exp(2\alpha t')x'^2\right)\right)\right)\end{aligned} \tag{71}$$

This expression agrees perfectly with the result obtained in [19] by using path integrals. Here also, it is verified that the result (71) is "gauge invariant".

## 6 Forced damped harmonic oscillator

In this section, we will apply the linear canonical transformation to the Hamiltonian of a damped harmonic oscillator, forced by an arbitrary function $h(t)$:

$$\hat{H} = \frac{\hat{p}^2}{2m(t)} + \frac{1}{2}m(t)\omega^2\hat{x}^2 - \exp(2\xi(t))h(t)\hat{x} \tag{72}$$

where

$$m(t) = m_0\exp(2\xi(t)) \tag{73}$$

and $\xi_(t)$ is a time-dependent function. The $g_i(t)$ functions in the generalized Hamiltonian (21) then take the forms

$$\begin{aligned} g_1(t) &= \frac{1}{2m_0\exp(2\xi(t))}, \\ g_2(t) &= \frac{1}{2}m_0\omega^2\exp(2\xi(t)), \\ g_6(t) &= -\exp(2\xi(t))h(t), \\ g_3(t) &= g_4(t) = g_5(t) = g_7(t) = 0 \end{aligned} \tag{74}$$

The suitable choice of the functions $f_i(t)$, solutions of differential equations (36) and (25), will enable us to determine the appropriate canonical transformation in order to calculate exactly the propagator of the forced damped harmonic oscillator. In this case system (25) and equations turn out to be

$$\begin{cases} \ddot{f}_1 + \left(-\ddot{\xi}(t) - \dot{\xi}^2(t) + \omega^2\right) f_1 = 0 \\ \ddot{f}_2 + 2\dot{\xi}(t)\dot{f}_2 + \omega^2 f_2 = \dfrac{h(t)}{m_0} \\ f_3 = m_0\exp(2\xi(t))\dot{f}_2 \end{cases} \tag{75}$$

The solution of these differential equations depends on the functions $h(t)$ and $\xi(t)$. However, the propagator of the forced damped harmonic oscillator is expressed as

$$\begin{aligned}\langle x'', t'' | x', t' \rangle &= \left(\frac{2\pi i\hbar f_1' f_1''}{m_0\exp\left(\xi''+\xi'\right)}\int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2(t)}\right)^{-1/2} \\ &\times \exp\left(\frac{im_0}{2\hbar}\left(\left(\frac{(x''-f_2'')}{f_1''\exp\left(-\xi''\right)} - \frac{(x'-f_2')}{f_1'\exp\left(-\xi'\right)}\right)^2 \Big/ \int_{t'}^{t''}\frac{\mathrm{d}t}{f_1^2(t)}\right.\right. \\ &\times \frac{2}{m_0}\left(f_3''x'' - f_3'x'\right) + \left(\frac{\dot{f}_1''}{f_1''} - \dot{\xi}(t'')\right)\left(x''-f_2''\right)^2 \\ &\left.\left.\times \exp 2\xi'' - \left(\frac{\dot{f}_1'}{f_1'} - \dot{\xi}(t')\right)\left(x'-f_2'\right)^2\exp 2\xi' - \frac{2}{m_0}\Lambda\right)\right)\end{aligned} \tag{76}$$

where

$$\begin{aligned}\Lambda = \int_{t'}^{t''}&\left(\frac{f_3^2}{2m_0\exp(2\xi(t))} + \frac{1}{2}m_0\omega^2\right. \\ &\left.\times \exp(2\xi(t))f_2^2 - h\exp(2\xi(t))f_2 + f_2\dot{f}_3\right)\mathrm{d}t\end{aligned} \tag{77}$$

The same remark about the "gauge invariance" is also valid here.

### 6.1 Particular cases

As a particular case, and to verify the result above, we will consider the following simple cases.

#### 6.1.1 A particle in a constant external field $F$

The Hamiltonian of the particle takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} - F\hat{x} \tag{78}$$

where $m$ is a constant and $\xi(t)$ and $\omega$ in (72) are null. Consequently, differential equations (75) become

$$\begin{cases} \ddot{f}_1 = 0 \\ \ddot{f}_2 = \dfrac{F}{m} \\ f_3 = m\dot{f}_2 \end{cases} \tag{79}$$

Hence, the solutions $f_i(t)$ take the forms

$$\begin{cases} f_1 = A_1 t + A_2 \\ f_2 = \dfrac{Ft^2}{2m} + B_1 t + B_2 \\ f_3 = Ft + mB_1 \end{cases} \tag{80}$$

which permits us to obtain the propagator in the case of a particle in a constant external field as

$$\begin{aligned} \langle x'', t'' | x', t' \rangle_0 &= \left( \frac{2\pi i\hbar(t''-t')}{m} \right)^{-1/2} \\ &\times \exp\left( \frac{i}{\hbar}\left( \frac{m(x''-x')^2}{2(t''-t')} + \frac{1}{2}F(t''-t')(x''+x') \right.\right. \\ &\left.\left. - \frac{F^2(t''-t')^3}{24m} \right)\right) \end{aligned} \tag{81}$$

The same result was obtained using a path-integral method [20]. Notice that the result (81) is "gauge invariant", i.e., independent of the choice of the constants.

#### 6.1.2 A simple harmonic oscillator driven by an external force

If we consider the function $\xi(t)$ in (72) to be null, we obtain the Hamiltonian of a simple harmonic oscillator driven by an external force as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - F(t)\hat{x} \tag{82}$$

Hence, differential equations (75) are written as

$$\begin{cases} \ddot{f}_1 + \omega^2 f_1 = 0 \\ \ddot{f}_2 + \omega^2 f_2 = \dfrac{F(t)}{m} \\ f_3 = m\dot{f}_2 \end{cases} \tag{83}$$

However, the solutions $f_i(t)$ take the forms

$$\begin{cases} f_1 = A_1 \cos\omega t + A_2 \sin\omega t \\ f_2 = \displaystyle\int_{t'}^{t} \frac{F(s)}{m\omega} \sin\omega(t-s)\,\mathrm{d}s + B_1\cos\omega t + B_2 \sin\omega t \\ f_3 = \displaystyle\int_{t'}^{t} F(s)\cos\omega(t-s)\,\mathrm{d}s - m\omega B_1 \sin\omega t + m\omega B_2 \cos\omega t \end{cases} \tag{84}$$

The replacement of these functions in (76) enables us to obtain the propagator in the case of a simple harmonic oscillator driven by an external arbitrary force as

$$\begin{aligned} \langle x'', t'' | x', t' \rangle &= \left( \frac{m\omega}{2\pi i\hbar \sin\omega T} \right)^{1/2} \\ &\times \exp\left( \frac{im\omega}{2\hbar\sin\omega T} \left( \cos\omega T(x''^2 + x'^2) - 2x''x' \right.\right. \\ &+ \frac{2x''}{m\omega}\int_{t'}^{t''} F(t)\sin\omega(t-t')\,\mathrm{d}t + \frac{2x'}{m\omega}\int_{t'}^{t''} F(t)\sin\omega(t''-t)\,\mathrm{d}t \\ &\left.\left. - \frac{2}{m^2\omega^2}\int_{t'}^{t''}\int_{t'}^{t} F(t)F(s)\sin\omega(t''-t)\sin\omega(s-t')\,\mathrm{d}s\ \mathrm{d}t \right)\right) \end{aligned} \tag{85}$$

where $T = t'' - t'$. The same result was obtained in [20] using a path-integral method.

We suppose now the frequency to be null: $\omega = 0$. The Hamiltonian then has the form

$$\hat{H} = \frac{\hat{p}^2}{2m} - F(t)\hat{x} \tag{86}$$

Therefore, differential equations (75) become

$$\begin{cases} \ddot{f}_1 = 0 \\ \ddot{f}_2 = \dfrac{F(t)}{m} \\ f_3 = m\dot{f}_2 \end{cases} \tag{87}$$

The following solutions verify these equations:

$$\begin{cases} f_1 = A_1 t + A_2 \\ f_2 = \displaystyle\int_{t'}^{t} \frac{F(s)}{m}(t-s)\,\mathrm{d}s + B_1 t + B_2 \\ f_3 = \displaystyle\int_{t'}^{t} F(s)\,\mathrm{d}s + mB_1 \end{cases} \tag{88}$$

yielding to the propagator of a particle in an arbitrary external force:

$$\begin{aligned} \langle x'', t'' | x', t' \rangle &= \left( \frac{m}{2\pi i\hbar T} \right)^{1/2} \exp\left( \frac{im}{2\hbar T} \left( (x''-x')^2 \right.\right. \\ &+ \frac{2x''}{m}\int_{t'}^{t''} F(t)(t-t')\,\mathrm{d}t + \frac{2x'}{m}\int_{t'}^{t''} F(t)(t''-t)\,\mathrm{d}t \\ &\left.\left. - \frac{2}{m^2}\int_{t'}^{t''}\int_{t'}^{t} F(t)F(s)(t''-t)(s-t')\,\mathrm{d}s\ \mathrm{d}t \right)\right) \end{aligned} \tag{89}$$

It is easily checked that the propagator (85) tends to the expression (89) of a particle in an arbitrary external force for $\omega = 0$.

The results (85) and (89) are effectively "gauge invariant".

## 7 Conclusion

In this work, we applied the Schwinger action principle to the case of general 1D time-dependent quadratic systems, with $\hat{x}'$ and $\hat{p}'$ additional terms, via quantum canonical transformations, without solving the operatorial Heisenberg equations of motion. To make the generalization, we found that in resolving certain differential equations, one obtains the temporal factors that must be used in linear canonical transformations, which makes it possible to transform the problem into that of a free particle overlapping all the particular cases of 1D quadratic systems. The final formula of the propagator is explicitly dependent on initial and final coordinates, and the time dependence is determined by the solutions of the differential equations.

Our result was used to calculate the propagator for the 1D quadratic system. In this step we have to solve only one differential equation in order to choose the suitable canonical transformation for this case. To check the efficiency of this method, the particular cases of a harmonic oscillator with a constant frequency, the harmonic oscillator with a time-dependent frequency, and the harmonic oscillator with time-dependent mass and frequency were then easily evaluated. To verify this result, we made two applications—the damped harmonic oscillator and the Calidora–Kanai oscillator—and found that the result agreed perfectly with that found in the literature obtained by means of path integrals. These calculations represent a basic step toward a generalization of the Schwinger action principle via quantum canonical transformations to the case of multidimensional quadratic systems such as coupled systems and to the interesting spinorial systems. The problem is under consideration.

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